

# The Master Field in Generalised $QCD_2$

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As an illustration of the formalism of the master field we consider generalised  $QCD_2$ . We show how Wilson Loop averages for an arbitrary contour can be computed explicitly and with some ease. A generalised Hopf equation is shown to govern the behaviour of the eigenvalue density of Wilson loops. The collective field description of the theory is therefore deduced. Finally, the non-trivial master gauge field and field strengths are obtained. These results do not seem easily accessible with conventional means.

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## 1. Introduction

We are faced with a paucity of techniques for understanding, quantitatively, the dynamics of strongly interacting field theories. One of the most intriguing, and probably the least well understood, proposals has been to study the theory in the so called large  $N$  limit. Here  $N$  refers to the size of the gauge group which may be, say,  $SU(N)$  or  $SO(N)$ . It is well known that these theories do simplify in the large  $N$  limit. The free energy and correlation functions have a well defined expansion in powers of  $\frac{1}{N}$ . Moreover the leading term — the planar limit — is thought to capture the essential physics.

One of the intriguing features of this limit is the existence of a master field [1]. This notion arises from the observation that the large  $N$  limit is in some sense a classical limit with  $\frac{1}{N}$  playing the role of  $\hbar$ . The factorisation of products of gauge invariant observables, with corrections of  $O(1/N)$  bears this out [2]. The master field refers to this “classical” configuration that dominates the path integral. Its knowledge would enable one to compute gauge invariant quantities without performing the functional integral. We simply evaluate them at this point in field space. In fact, it can be argued that by a suitable gauge transformation the master gauge field  $\bar{A}_\mu(x)$  can be made space time independent. Thus, for example, in  $QCD_4$  we need obtain just four “ $\infty \times \infty$ ” matrices  $\bar{A}_\mu$ !

Tantalising as this prospect may seem, it has proved rather elusive to work with. We need to understand what these “ $\infty \times \infty$ ” matrices really are. And then again we need effective ways to compute them.

Recently, it has been possible to get a better hold on this concept [3], [4] [5]. The mathematical formalism of non-commutative probability theory [6] seems to be the right setting for the master field. Let us briefly summarise the relevant facts.

The master field can be thought of as an operator living in a Fock space generated by creation operators obeying no relations. In other words, a Fock space spanned by the states

$$(\hat{a}_{i_1}^\dagger)^{n_{i_1}} (\hat{a}_{i_2}^\dagger)^{n_{i_2}} \dots (\hat{a}_{i_k}^\dagger)^{n_{i_k}} |\Omega\rangle \quad (1.1)$$

where

$$\hat{a}_i |\Omega\rangle = 0, \quad \hat{a}_i \hat{a}_j^\dagger = \delta_{ij}. \quad (1.2)$$

Here the subscripts  $i$  can take either discrete or continuous values. In an  $n$ -matrix model  $i$  will run from 1 to  $n$ . A general operator, the master field included, will be built out of these  $\hat{a}_i$ ’s and  $\hat{a}_j^\dagger$ ’s.

In the theory of non-commutative probability, a very special place is occupied by the so called free random variables. They are akin to independent random variables in usual probability theory. They are best thought of as the large  $N$  limit of matrices with independent distributions. Thus for an independent  $n$ -matrix model the master fields  $\hat{M}_i$  corresponding to the matrices  $M_i$  are free random variables. They turn out to have a simple realisation on this fock space.

$$\hat{M}_i = \hat{a}_i + \sum_{n=0}^{\infty} M_{n+1}^{(i)} \hat{a}_i^{\dagger n}. \quad (1.3)$$

An arbitrary invariant correlation function of the theory is then computable using

$$\langle \text{Tr} [M_{i_1}^{n_1} M_{i_2}^{n_2} \dots M_{i_k}^{n_k}] \rangle = \left\langle \Omega | \hat{M}_{i_1}^{n_1} \hat{M}_{i_2}^{n_2} \dots \hat{M}_{i_k}^{n_k} | \Omega \right\rangle. \quad (1.4)$$

It turns out that the  $M_n^{(i)}$  have a rather simple physical interpretation. They are determined solely by the distribution for  $M_i$  and are the connected  $n$  point Green's functions. In other words, dropping the subscript  $i$ ,

$$M_n = \langle \text{tr}[M^n] \rangle_{\text{conn.}}. \quad (1.5)$$

Or equivalently [3] the generating function for connected Green's functions

$$zM(z) = 1 + \sum_{n=1}^{\infty} M_n z^n \quad (1.6)$$

is the inverse function of the resolvent

$$R(\zeta) = \sum_{n=0}^{\infty} \zeta^{-n-1} \langle \Omega | M^n | \Omega \rangle. \quad (1.7)$$

A generalisation of eqn.(1.3) to an arbitrary matrix model also exists. We refer the reader to [3] for details on this as well as equations of motion, examples and more.

However, it has been difficult to compute the master field in this formalism other than in cases where the large  $N$  correlation functions have already been obtained by conventional methods such as the saddle point or the Schwinger-Dyson equations. It might then appear that our framework merely allows us to rewrite results of the large  $N$  theory in a compact way. This is not true. As we shall see in this paper the example of generalised  $QCD_2$  is an instance where we can profitably apply ideas from the theory of free random variables to explicitly compute many quantities of interest. Due to the nonlinearity of the theory,

the usual methods, mentioned earlier, do not appear very tractable. The main notion that is exploited here is that of a multiplicative free family of random variables [6],[3]— a notion that we will elaborate on below. It allows us to construct the Wilson loop average for an arbitrary contour from knowledge of certain infinitesimal ones. We also obtain a realisation of the generalised Hopf equation as a collective field equation for the theory. This will allow us to deduce the collective field theory in the general case of the cylinder. Finally, one can construct the master field strength and gauge field.

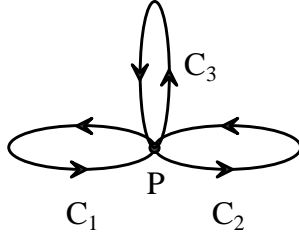
Generalised  $QCD_2$  [7], [8] has been argued to be a theory of interest in the search for potential string theory descriptions of higher dimensional  $QCD$ . It generalises the heat kernel action of  $QCD_2$ . Alternatively, it can be defined as the general perturbation to topological  $YM_2$ :

$$\mathcal{Z} = \int [\mathcal{D}A][\mathcal{D}B] \exp(- \int (\text{tr}(iBF) - \Phi(B))). \quad (1.8)$$

While it doesn't have propagating modes, it is nevertheless more complex in behaviour than ordinary  $QCD$ . For instance, it exhibits the large  $N$  phase transition [9] on Riemann surfaces of almost all genera [10]. It therefore seems worthwhile to have at hand, explicit results analogous to [11]. From this point of view the generalised Hopf equation that we obtain below would have bearing on a world sheet lagrangian description. In Section 2 we outline the notion of multiplicatively free families and its appearance in  $QCD_2$  where it enables us to compute arbitrary loop averages. We proceed in Section 3 to calculate the loop averages  $\langle \text{tr}[U^n] \rangle$  for a simple loop which is then sufficient to obtain a general one. In Section 4 we show that the behaviour of eigenvalues of loops are governed by the generalised Hopf equation. The collective field theory is then reconstructed. Section 5 is devoted to the construction of the master gauge field and field strength. Discussions and conclusions comprise section 6. An appendix bears the burden of a necessary combinatorial calculation.

## 2. Loops as Free Random Variables

We indicate how a knowledge of infinitesimal loops in generalised  $QCD_2$  (ordinary  $QCD_2$  is a particular case) can be used to compute arbitrary loop averages. The basic loop entities that we will be working with are the simple loops. (By a simple loop we shall henceforth mean a non-self intersecting contour.) Their importance lies in the fact that a set of simple loops, based at one point and non-overlapping, correspond to a family of free random variables ( Fig. 1).



**Fig. 1:** Three Simple Loops

To see this, notice that we could work with the heat kernel action for generalised  $QCD_2$  [8],

$$\mathcal{Z} = \int \prod_L \mathcal{D}U_L \prod_{\text{plaquettes}} Z_P[U_P]; \quad (2.1)$$

$$Z_P[U_P] = \sum_R d_R \chi_R[U_P] e^{-\Lambda(R)A_P},$$

where  $\Lambda(R)$  is an arbitrary function of the Casimirs. Now the self similar nature of the heat kernel always allows us to triangulate the plane such that the contours  $C_i$  are the borders of some of the triangles. Let us denote the  $U(N)$  holonomies along  $C_i$  by  $U_i$ . Then in any computation of averages of products of  $U_i$ , we can integrate out (only on the plane) all other link variables. We are thus left with an equivalent measure  $\mathcal{Z} = \int \prod_i \mathcal{D}U_i \prod_i Z[U_i]$ , where the product runs over all the simple loops,  $U_i$ . In other words,  $U_i$  have independent distributions.

Actually, the  $U_i$ 's are not just free random variables. They comprise what is known as a multiplicative free family [6]. Briefly what this means is the following: The product of two free random variables with distributions,  $\mu_1$  and  $\mu_2$  is also a free random variable with some distribution  $\mu_3$  denoted by  $\mu_1 \otimes \mu_2$ . A one parameter family of free random variables, such that  $\mu_{t_1} \otimes \mu_{t_2} = \mu_{t_1 t_2}$ , will be called a multiplicative free family. (Or equivalently  $\mu_{s_1} \otimes \mu_{s_2} = \mu_{s_1 + s_2}$ , if we redefine the parameter  $t \rightarrow s = \log t$ .)

Here the area plays the role of the parameter  $s$ . In other words, for the two simple loops  $C_1$  and  $C_2$  (fig. 1),  $\hat{U}_{C_1}(A_1)\hat{U}_{C_2}(A_2)$  has the same distribution as  $\hat{U}_{C_1 \circ C_2}(A_1 + A_2)$ . This follows from the self reproducing nature of the heat kernel action (2.1) together with its exponential dependence on the area. A more direct argument is made in [3].

This allows us to obtain the distribution for a simple loop of finite area by starting from one of infinitesimal area. The precise way to do this is to use the S-Transform. This is

analogous to the Mellin transform of ordinary probability theory, i.e. it is multiplicative for the product of distributions. Therefore for our multiplicative free family,  $S_{A_1}(z)S_{A_2}(z) = S_{A_1+A_2}(z)$ . It then follows that  $S_A(z)$  is exponential in  $A$ .

The function  $S(z)$  for a non-commutative random variable  $U$  is constructed as follows [6]: If

$$\phi(j) = \sum_{n=1}^{\infty} \langle \Omega | \hat{U}^n | \Omega \rangle j^n, \quad (2.2)$$

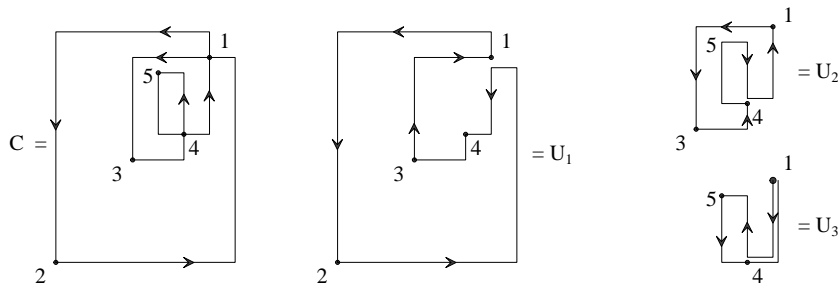
then construct the inverse function  $\chi(z)$ , i.e.  $\phi(\chi(z)) = z$ . The S-transform is defined as:

$$S(z) = \frac{1+z}{z} \chi(z). \quad (2.3)$$

In the next section we shall compute  $S_A(z)$  for infinitesimal  $A$  and then exponentiate it to obtain that for finite  $A$ . Then we shall take the ‘inverse transform’ to finally arrive at  $\langle \text{tr}[U^n] \rangle$ .

We conclude the section with a sketch of how arbitrary loop averages are computed. This requires two ingredients. Firstly, a geometrical decomposition of an arbitrary loop into a ‘word’ built out of simple, non-overlapping loops, based at one point. And secondly, that the latter are free random variables.

The first was illustrated at length in [3]. Here we shall merely give an instance of how it works.



**Fig. 2:** A Loop and its Decomposition into Simple Loops

With the labelling in fig. 2 we can write the contour as

$$\begin{aligned}
C &= (12)(\overline{21})(13)(34)(45)(\overline{54})(41) \\
&= \underbrace{(12)(\overline{21})(14)(43)(31)}_{U_1} (13)(34)(41)(13)(34)(45)(\overline{54})(41) \\
&= \underbrace{(12)(\overline{21})(14)(43)(31)}_{U_1} \underbrace{(13)(34)(\overline{45})(54)(41)}_{U_2} \underbrace{(14)(45)(\overline{54})(41)}_{U_3} \\
&\quad \underbrace{(13)(34)(\overline{45})(54)(41)}_{U_2} \underbrace{(14)(45)(\overline{54})(41)}_{U_3} \underbrace{(14)(45)(\overline{54})(41)}_{U_3} \\
&= U_1 U_2 U_3 U_2 U_3^2.
\end{aligned} \tag{2.4}$$

The main strategy is to introduce backtracking or ‘thin’ loops to peel off the simple loops  $U_i$  corresponding to the various windows.

Having decomposed the loop into a word, we can exploit the fact that the  $U_i$  are free. The loop average for (2.4), for instance, reads as

$$\begin{aligned}
W(C) &= \langle \text{tr}[U_1 U_2 U_3 U_2 U_3^2] \rangle = \langle \text{tr} U_1 \rangle \langle \text{tr}[U_2 U_3 U_2 U_3^2] \rangle \\
&= \langle \text{tr} U_1 \rangle (\langle \text{tr} U_2 \rangle^2 \langle \text{tr} U_3^3 \rangle - \langle \text{tr} U_2 \rangle^2 \langle \text{tr} U_3 \rangle \langle \text{tr} U_3^2 \rangle \\
&\quad + \langle \text{tr} U_2^2 \rangle \langle \text{tr} U_3 \rangle \langle \text{tr} U_3^2 \rangle).
\end{aligned} \tag{2.5}$$

Here we have used a recursion relation for free random variables (see [3]) to factorise the word into ‘letters’ of the form  $\langle \text{tr}[U^n] \rangle$ . Once we evaluate the latter, we are done.

### 3. $\langle \text{tr}[U^n] \rangle$ in Generalised $QCD_2$

As mentioned earlier, we utilise the S-Transform (2.3),(2.2). This requires us to evaluate  $\langle \text{tr}[U^n] \rangle$  for an infinitesimal loop. Again it is the heat kernel definition that is useful. By definition

$$W_n \equiv \sum_R d_R \int \mathcal{D}U \chi_R(U) \text{tr}[U^n] e^{-\frac{\Lambda}{2} \Lambda(R)}. \tag{3.1}$$

To evaluate the  $U$  integral we can use the fermion representation of  $U(N)$  [12]. The characters are now the Slater fermion wavefunctions. In a second quantised language  $\text{tr}[U^n]$  becomes a fermion bilinear and

$$\int \mathcal{D}U \chi_R(U) \text{tr}[U^n] = \langle O | \sum_m B_{n-m}^\dagger B_m | R \rangle. \tag{3.2}$$

Here  $|O\rangle$  is the ground state of the  $N$  fermions and *not* the vacuum. In terms of Young tableaux this is the state with no boxes.  $|R\rangle$  is the fermionic state labelled by integers  $\{n_i\}$ . These integers are related to the number of boxes  $h_i$  in the  $i$ th row of the young tableau by  $n_i = (N - 1)/2 + 1 - i + h_i$ .

It is now easy to see that the only representations that contribute in (3.2) have

$$\begin{aligned} h_1 &= n - m \\ h_i &= 1 \quad (2 \leq i \leq m + 1) \end{aligned} \tag{3.3}$$

with  $m$  any integer between 0 and  $n - 1$ . These come with a relative weight of  $(-1)^m$ . Therefore

$$W_n = \sum_{m=0}^{n-1} (-1)^m d_{R_m} e^{-\frac{A}{2} \Lambda(R_m)}. \tag{3.4}$$

Our task is simplified in that we need only the large  $N$  limit of (3.4) and infinitesimal area. In other words, the sum

$$W_n(\Delta A) = \sum_{m=0}^{n-1} (-1)^m d_{R_m} \left(1 - \frac{\Delta A}{2} \Lambda(R_m)\right). \tag{3.5}$$

Now  $\Lambda(R)$  could, in general, take the form [8]

$$\Lambda(R) = \sum_{\{k_i\}} a_{\{k_i\}} C_{\{k_1+2k_2+3k_3+\dots\}}(R) \tag{3.6}$$

where  $C_{\{k_1+2k_2+3k_3+\dots\}}(R)$  is a generalised Casimir. In the continuum theory this corresponds to (1.8) with

$$\Phi(B) = \sum_{\{k_i\}} a_{\{k_i\}} \prod_i (\text{tr} B^i)^{k_i}. \tag{3.7}$$

For reasons of clarity and computational efficacy we'll henceforth restrict ourselves to the sufficiently general case

$$\Lambda(R) = \sum_k \frac{a_k}{N^{k-1}} C_k(R) \tag{3.8}$$

i.e. the higher Casimirs. In the path integral this translates into

$$\Phi(B) = \sum_k a_k \text{tr} B^k. \tag{3.9}$$

The considerations generalise in a straightforward manner, however.



We can perform the sum (3.5) with  $\Lambda(R_m)$  given by (3.8). This is undertaken in the appendix. The result is

$$\frac{1}{N}W_n(\Delta A) = 1 - \frac{\Delta A}{2} \sum_k a_k n \binom{n+k-2}{k-1} \quad (3.10)$$

The S-transform is now readily computed. From (2.2)

$$\begin{aligned} \phi(j) &= \sum_{n=1}^{\infty} \left(1 - \frac{\Delta A}{2} \sum_k a_k n \binom{n+k-2}{k-1}\right) j^n \\ &= \frac{j}{1-j} \left(1 - \frac{\Delta A}{2} \sum_k a_k \frac{1}{(1-j)^k} (1 + (k-1)j)\right). \end{aligned} \quad (3.11)$$

According to (2.3) we need the inverse to  $\phi(j)$ . This is accomplished easily since we only need it to lowest order in  $\Delta A$ .

$$\chi_{\Delta A}(z) = \frac{z}{1+z} \left(1 + \frac{\Delta A}{2} \sum_k a_k (1+kz)(1+z)^{k-2}\right). \quad (3.12)$$

As argued earlier,  $S_A(z)$  for finite area is obtained by exponentiating  $S_{\Delta A}(z)$ .

$$\begin{aligned} S_A(z) &= e^{\frac{A}{2}\varphi(z)} \\ \varphi(z) &\equiv \sum_k a_k (1+kz)(1+z)^{k-2} \\ &= \sum_k k(a_k - a_{k+1})(1+z)^{k-1} \end{aligned} \quad (3.13)$$

We are now in a position to calculate  $\langle \text{tr}[U^n] \rangle$ , given the explicit form of (3.13). From (2.2), we conclude that

$$\langle \text{tr}[U^n] \rangle = \frac{1}{2\pi} \int \phi(e^{-i\theta}) e^{in\theta} d\theta. \quad (3.14)$$

We make the change of variables  $\chi(z) = e^{-i\theta}$  exploiting the fact that  $\phi(\chi(z)) = z$ . Finally after an integration by parts we reach

$$\begin{aligned} \langle \text{tr}[U^n] \rangle &= \frac{1}{n} \oint \frac{dz}{2\pi i} [\chi_A(z)]^{-n} \\ &= \frac{1}{n} \oint \frac{dz}{2\pi i} \left(1 + \frac{1}{z}\right)^n e^{-n\frac{A}{2} \sum_k a_k (1+kz)(1+z)^{k-2}}. \end{aligned} \quad (3.15)$$

This is the expression that generalises the Laguerre polynomials that appeared in the usual  $QCD_2$  [13]. They are, of course, recovered by restricting ourselves to the case with

$a_k = \delta_{k,2}$ . We also note that the  $a_1$  dependence is only through a multiplicative factor of  $e^{-na_1 \frac{A}{2}}$ . The first few averages are

$$\begin{aligned} \langle \text{tr} U \rangle &= e^{-\frac{A}{2} \sum_k a_k}; \quad \langle \text{tr}[U^2] \rangle = (1 - A \sum_k a_k(k-1)) e^{-2\frac{A}{2} \sum_k a_k} \\ \langle \text{tr}[U^3] \rangle &= (1 - 3\frac{A}{4} \sum_k a_k(k-1)(k+2) + 3\frac{A^2}{2} [\sum_k a_k(k-1)]^2) e^{-3\frac{A}{2} \sum_k a_k}. \end{aligned} \quad (3.16)$$

The general structure is

$$\langle \text{tr}[U^n] \rangle = P_n(A) e^{-n\frac{A}{2} \sum_k a_k}, \quad (3.17)$$

where  $P_n(A)$  is a polynomial of degree  $n-1$  determined by

$$P_n(A) = \frac{1}{(n-1)!} \frac{d^{n-1}}{dx^{n-1}} [x^n e^{-n\frac{A}{2} \sum_k k(a_k - a_{k+1})(x^k - 1)}] \Big|_{x=1}. \quad (3.18)$$

As argued in the previous section this is all we need to compute the loop average of an arbitrary contour.

#### 4. The Generalised Hopf Equation and Collective Field Theory

The Hopf equation arises in the collective field theory description in  $QCD_2$  [12]1. It governs the behaviour of the eigenvalue density of Wilson loops [14]. In this case we shall find an analogous equation — the generalised Hopf equation [3]. We shall demonstrate that

$$F(\theta, A) \equiv \phi_A(e^{i\theta}) = e^{i\theta} R(e^{i\theta}, A) - 1 \quad (4.1)$$

obeys the equation

$$\frac{\partial F}{\partial A} + \frac{i}{2} \varphi(F) \frac{\partial F}{\partial \theta} = 0, \quad (4.2)$$

with  $\varphi(F)$  as in (3.13). Here  $R(e^{i\theta}, A)$  is the resolvent

$$R(\zeta, A) = \sum_{n=0}^{\infty} \langle \text{tr}[U(A)^n] \rangle \zeta^{-(n+1)}. \quad (4.3)$$

We also note that the eigenvalue density of loops  $\sigma(\theta, A)$  is given by

$$Re F(\theta, A) = \pi \sigma(\theta, A) - \frac{1}{2}. \quad (4.4)$$

If we define  $\theta(z, A)$  such that  $e^{i\theta(z, A)} = \chi_A(z)$ , then  $F(\theta(z, A), A) = z$ . This follows from (4.1) and the definition of  $\chi_A(z)$  as the inverse to  $\phi(j)$ . Therefore

$$\frac{dF(\theta(z, A), A)}{dA} \Big|_z = \frac{\partial F}{\partial A} \Big|_\theta + \frac{\partial F}{\partial \theta} \Big|_A \frac{\partial \theta(z, A)}{\partial A} \Big|_z = 0. \quad (4.5)$$

Knowing, as we do,  $\chi_A(z)$  for generalised  $QCD_2$ . we see that

$$\theta(z, A) = \frac{iA}{2} \varphi(z) - i \ln\left(\frac{1+z}{z}\right). \quad (4.6)$$

This, together with (4.5) enables us to arrive at (4.2).

Once again, the hopf equation of  $QCD_2$  is regained in the case  $a_k = \delta_{k,2}$  after a minor shift. Like the Hopf equation, (4.2) is integrable as well. In fact, it is easily checked that

$$F(\theta, A) = F_0\left(\theta - \frac{iA}{2} \varphi(F(\theta, A))\right) \quad (4.7)$$

is the (implicit) solution of this equation. Here  $F_0(\theta) = F(\theta, A = 0)$  is the necessary initial condition. The solution is also completely determined once we specify, say,  $\sigma(\theta, A = 0)$  and  $\sigma(\theta, A = \infty)$ .

One might imagine obtaining this equation from the analogue of the loop equations for generalised  $QCD_2$ . These highly non-linear equations would have to be derived from a series of Schwinger-Dyson equations for loops. This does not seem to be a very easy approach.

Alternatively, one might imagine a derivation of (4.2) from a collective field theory description [15] of generalised  $QCD_2$ . Logically, one would start from the fermionic description. The hamiltonian is a rather complicated polynomial in the operators  $\frac{\partial^m}{\partial \theta^m}$ . One would have to then bosonise them in their second quantised form and look at the classical equations of motion. Here, we have seen it arise much more simply.

In fact, though it was derived here for the plane, we can conclude that the equation will be true on the sphere, or more generally, the cylinder. This follows since these are local evolution equations which are sensitive to the global geometry only via the initial/final conditions. Thus, on the cylinder we would specify the boundary holonomies or eigenvalue densities  $\sigma(\theta, A = 0)$  and  $\sigma(\theta, A = A_0)$  ( $A_0$  is the area of the cylinder). Then  $\sigma(\theta, A)$  would be determined from (4.2) with  $F(\theta, A)$  a complex function such that

$$F(\theta, A) = \pi \sigma(\theta, A) - \frac{1}{2} - iv(\theta) \quad (4.8)$$

This can be interpreted as a general kind of fluid flow with density  $\sigma$  and velocity  $v$ .

But now we can turn the logic around and derive the collective field theory Hamiltonian whose classical equation of motion is (4.2). It is convenient to redefine variables to

$$G(\theta, A) = 1 + F(\theta, A) = \pi\sigma(\theta, A) + \frac{1}{2} - i\partial_\theta\Pi(\theta, A) \quad (4.9)$$

where  $\Pi$  is the canonical momentum conjugate to  $\sigma$ . Then we have the Poisson bracket

$$\{G(\theta, A), G(\theta', A)\} = -i2\pi\partial_\theta\delta(\theta - \theta'). \quad (4.10)$$

The generalised Hopf equation now reads as

$$\frac{\partial G}{\partial A} + \frac{i}{2} \sum_k k(a_k - a_{k+1}) G^{k-1} \frac{\partial G}{\partial \theta} = 0. \quad (4.11)$$

This is the Hamiltonian equation of motion

$$\begin{aligned} \frac{\partial G}{\partial A} &= \{H, G\} \\ H &= -\frac{1}{4\pi} \sum_k \frac{1}{k+1} (a_k - a_{k+1}) \int d\theta [G^{k+1}(\theta, A) + c.c.]. \end{aligned} \quad (4.12)$$

This is thus the Collective field hamiltonian which can be expressed in terms of  $\sigma(\theta, A)$  and  $\Pi(\theta, A)$ . Our considerations thus far have been in euclidean space. The minkowski version reads as

$$\begin{aligned} H_M &= \frac{1}{2} \sum_k \frac{1}{k+1} (a_k - a_{k+1}) \int d\theta [(\frac{1}{2} + P_+)^{k+1} + (\frac{1}{2} - P_-)^{k+1}] \\ P_\pm &= \partial_\theta \Pi(\theta, A) \pm \pi\sigma(\theta, A). \end{aligned} \quad (4.13)$$

The usual  $D = 1$  matrix model hamiltonian [15]

$$H = \frac{1}{2\pi} \int d\theta \sigma(\theta) \{ \partial_\theta \Pi(\theta)^2 + \frac{\pi^2}{3} \sigma(\theta)^2 \} \quad (4.14)$$

corresponds to taking  $a_k = \delta_{k,2}$ .

## 5. The Master Field

Given that we have computed arbitrary loop averages, we expect to have enough information to construct the master field strength. To actually do this we will first obtain master loop operators  $\hat{U}(\Delta A)$  for infinitesimal loops. These take the form

$$\begin{aligned}\hat{U} &= \hat{a} + \sum_{k=0}^{\infty} \omega_{k+1} \hat{a}^{\dagger k} \\ &< \text{tr}[U^n] > = \langle \Omega | \hat{U}^n | \Omega \rangle.\end{aligned}\tag{5.1}$$

where we need  $\omega_k$  only to lowest order in  $\Delta A$ . Since  $U(y) = \frac{1}{y} + \sum_{k=0}^{\infty} \omega_{k+1} y^k$  is the inverse of the resolvent, using (4.1) and (2.2) we have

$$R\left(\frac{1}{\chi(z)}\right) = (1+z)\chi(z) \Rightarrow U((z+1)\chi(z)) = \frac{1}{\chi(z)}.\tag{5.2}$$

Therefore (3.13) implies

$$\begin{aligned}U(y) &= \frac{1}{y}(1+z(y)); \quad y = ze^{\frac{A}{2}\varphi(z)} \\ &\Rightarrow \sum_{k=0}^{\infty} \omega_{k+1} y^{k+1} = z(y)\end{aligned}\tag{5.3}$$

This is easily solved to lowest order in  $A$  to obtain

$$z(y) = y\left(1 - \frac{\Delta A}{2}\varphi(y)\right).\tag{5.4}$$

So the master loop operator  $\hat{U}(\Delta A)$  is

$$\hat{U}(\Delta A) = \hat{a} + 1 - \frac{\Delta A}{2}\varphi(\hat{a}^\dagger).\tag{5.5}$$

Let us now construct the operator  $\hat{H}$  such that  $e^{i\hat{H}} = \hat{U}$ . This is best done as follows.

$$\hat{U}^n = e^{in\hat{H}} = 1 + in\hat{H} - \frac{n^2}{2}\hat{H}^2 + \dots\tag{5.6}$$

Therefore  $\hat{H}$  is identified as the linear term in  $n$  in  $\hat{U}^n$ . Again matters are simplified in having to keep only terms of  $O(\Delta A)$ . Then from (5.5)

$$\begin{aligned}\hat{U}^n &= (\hat{a} + 1 - \frac{\Delta A}{2}\varphi(\hat{a}^\dagger))^n \\ &= (\hat{a} + 1)^n - \frac{\Delta A}{2} \sum_{m=0}^{n-1} (\hat{a} + 1)^m \varphi(\hat{a}^\dagger) (\hat{a} + 1)^{n-1-m}\end{aligned}\tag{5.7}$$

First look at the term of order  $\Delta A$ . Terms in this expansion of the form  $\hat{a}^{\dagger l} \hat{a}^m$  do not contribute to  $\hat{H}^n$  in leading order in  $\Delta A$ . We group the other terms where  $\hat{a}^m$  multiplies on the left and look for the contribution linear in  $n$ . This gives us the term

$$\sum_{r=0} \frac{(-1)^r}{(r+1)} \hat{a}^r \varphi(\hat{a}^\dagger). \quad (5.8)$$

In this expression we can drop the terms purely of the form  $\hat{a}^m$ . Since there are already such terms of  $O(1)$  and the  $O(\Delta A)$  only contribute to higher order. The  $O(1)$  terms can be easily read from the  $A = 0$  limit when  $\hat{U} = \hat{a} + 1$ . Putting it all together

$$\begin{aligned} \hat{H} &= -i(\log(1 + \hat{a})) \left\{ 1 + \frac{\Delta A}{2} \hat{a}^\dagger \varphi(\hat{a}^\dagger) \right\} \\ &= \sum_{r=1}^{\infty} \frac{(-1)^r}{r} \hat{a}^r + \frac{\Delta A}{2} \sum_n c_n \hat{a}^{\dagger n} \\ c_n &= \sum_k a_k \sum_{r=0}^{k-n-1} \frac{(-1)^r}{(r+1)} (n+r+1) \binom{k-1}{n+r} \end{aligned} \quad (5.9)$$

We notice that  $c_0 = 0$ .

Matters are simplified if we choose axial gauge  $A_1 = 0$ . Then the holonomy for an infinitesimal loop becomes

$$U = P \exp(i \oint_C A_\mu dx^\mu) = \exp(i \partial_1 A_0 \Delta A). \quad (5.10)$$

We can then identify the large  $N$  limit of the two matrices:

$$\hat{F} \Delta A = \partial_1 A_0 \Delta A = \hat{H} \quad (5.11)$$

with  $\hat{H}$  as in (5.9). Since  $c_0 = 0$ ,  $\hat{F}$  has a vanishing 1-point function which is expected. The presence of the  $\Delta A$  is a consequence of discretisation of the action. This ensures that correlation functions of the field strength  $F$  do not have delta functions in them. In this gauge the master field  $\hat{A}_1 = 0$  and  $\hat{A}_0$  is determined by (5.11) and (5.9).

## 6. Discussions and Conclusions

In this work we have applied the formalism of the master field to explicitly compute the large  $N$  limit of many quantities in generalised  $QCD_2$ . This was meant to illustrate the utility of the framework of non-commutative probability theory in actual computations.

As we have seen, it is not easy to see how to obtain the collective field theory description or the expressions for loop averages and gauge fields with the usual means available to us. It is worthwhile to make some remarks regarding them here: The collective field equations of motion of Section 4 are related to the non-commutative probability distributions in the same way as the heat equation is related to the Gaussian distribution. This gives us an inkling as to the mathematical relation of the collective field description and the underlying matrix model. It might help to understand the relation better so as to arrive at tractable collective field descriptions of realistic theories. On a technical level, we note the simplification gained in going to an operator construction of the master field. In Section 5 we were able to go from loop averages to gauge fields, simply by performing standard operator manipulations like taking the logarithm. Analogous operations in a large  $N$  matrix description do not appear easy.

We have, of course, in this paper, exploited a very special property of this system, namely that of being multiplicatively free. This would not straightforwardly generalise to higher dimensions. Nevertheless, the lesson that is perhaps to be drawn is that techniques can be developed in the operator framework for the master field which can take us beyond conventional approaches.

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## Appendix

Here we undertake to perform the sum

$$W_n(\Delta A) = \frac{1}{N} \sum_{m=0}^{n-1} (-1)^m d_{R_m} \left(1 - \frac{\Delta A}{2} \Lambda(R_m)\right) \quad (6.1)$$

to leading order in large  $N$ , over representations  $R_m$  of (3.3). The dimension of the representation is given by

$$d_{R_m} = \prod_{i>j} \left(1 + \frac{h_j - h_i}{i - j}\right) = \frac{1}{n} \frac{(N + n - m - 1)!}{(N - m - 1)! m! (n - m - 1)!}. \quad (6.2)$$

$\Lambda(R_m)$  is given by (3.8), and the higher Casimirs are in general given by

$$C_k(R) = \sum_{i=1}^N l_i^k \gamma_i, \quad (6.3)$$

$$l_i = h_i + N - 1; \quad \gamma_i = \prod_{i \neq j} \left(1 - \frac{1}{l_i - l_j}\right).$$

In our case (3.3), this can be computed to be

$$\begin{aligned} C_k(R_m) &= l_1^k \gamma_1 + l_{m+1}^k \gamma_{m+1} \\ &= \frac{n}{n-1} [(N+n-m-1)^{k-1} (n-m-1) + (N-m)^{k-1} m] \\ &= \frac{n}{n-1} [(N+n-m-1)^k - (N-m)^k - N(k \rightarrow k-1)]. \end{aligned} \quad (6.4)$$

Using (6.2) it is easy to show that the  $O(1)$  term in (6.1) is indeed 1. The non-trivial part is the  $O(\Delta A)$  term which reads

$$\begin{aligned} \sum_{k>0} \frac{a_k}{N^k} \frac{n}{n-1} \sum_{m=0}^{n-1} (-1)^m \binom{N+n-m-1}{n} \binom{n-1}{m} \\ \times [(N+n-m-1)^k - (N-m)^k - N(k \rightarrow k-1)]. \end{aligned} \quad (6.5)$$

For fixed  $k$  consider the  $(N-m)^k$  term. Expanding in powers of  $N$ , the sum can be expressed as the coefficient of  $y^{N-1}$  in

$$\begin{aligned} \frac{n}{n-1} \sum_{r=0}^k (-1)^r \binom{k}{r} N^{k-r} \sum_{m=0}^{n-1} (-1)^m \binom{n-1}{m} (1+y)^{N+n-m-1} \left(z \frac{d}{dz}\right)^r z^m \Big|_{z=y, y^{N-1}} \\ = (-1)^n \frac{n}{n-1} \sum_{r=0}^k (-1)^r \binom{k}{r} N^{k-r} (1+y)^N \left(z \frac{d}{dz}\right)^r (z-y-1)^{n-1} \Big|_{z=y, y^{N-1}}. \end{aligned} \quad (6.6)$$

Next we write  $(z \frac{d}{dz})^r = \sum_{l=1}^r C_l^r z^l \frac{d^l}{dz^l}$ , where  $C_l^r$  are coefficients determined by the recursion relation

$$C_l^r = l C_l^{r-1} + C_{l-1}^{r-1}; \quad C_r^r = 1. \quad (6.7)$$

Thus (6.6) reads as

$$\frac{n}{n-1} \sum_{r=0}^k (-1)^r \binom{k}{r} N^{k-r} (1+y)^N \sum_{l=1}^r C_l^r (-1)^l y^l (n-1)(n-2) \dots (n-l) \Big|_{y^{N-1}}. \quad (6.8)$$



Since to leading order in  $N$  we only need the term that goes as  $N^k$ , it suffices to look for the coefficient of  $N^r$  in

$$(1+y)^N y^l|_{y^{N-1}} = \binom{N}{l+1} = \frac{N^{l+1}}{(l+1)!} + \frac{N^l}{2(l-1)!} + O(N^{l-1}). \quad (6.9)$$

Thus only the  $l = r$  and  $l = r - 1$  terms contribute in the sum over  $l$  in (6.8). Therefore, using  $C_r^r = 1$  and  $C_{r-1}^r = \frac{r(r-1)}{2}$ , the leading contribution to (6.8) simplifies into

$$-\frac{n}{2}N^k \sum_{r=0}^k \binom{k}{r} \binom{n-1}{r-1} = -\frac{n}{2}N^k \binom{n+k-1}{n}. \quad (6.10)$$

It can be checked that the  $(N+n-m-1)^k$  term in (6.5) gives a contribution of opposite sign but same magnitude as (6.10). Therefore the full contribution from (6.5) reads as

$$\sum_{k>0} a_k n \left[ \binom{n+k-1}{n} - \binom{n+k-2}{n} \right] = \sum_{k>0} a_k n \binom{n+k-2}{n-1}. \quad (6.11)$$

This is the result used in (3.10).

## References

- [1] E. Witten, *in Recent Developments in Gauge Theories* eds. G. 'tHooft et. al. Plenum Press, New York and London (1980).
- [2] A. Migdal, *Ann. Phys.* **109**,365 (1977).
- [3] R.Gopakumar, D.J.Gross,*Nucl. Phys.* **B451**, 379 (1995)
- [4] M.R.Douglas hep-th/9409098, *Phys. Lett.* **344B**, 117, (1995)
- [5] I.Ya.Aref'eva,I.V.Volovich hep-th/9510210
- [6] D. V. Voiculescu, K.J. Dykema and A. Nica *Free Random Variables* AMS , Providence (1992)
- [7] M.R.Douglas, K.Li, M.Staudacher *Nucl. Phys.* **B420** 118, (1994)
- [8] O.Ganor, J.Sonnenschein, S.Yankielowicz, *Nucl. Phys.* **B434** 139, (1995)
- [9] M.R.Douglas,V.A.Kazakov *Phys. Lett.* **319B** 219, (1993)
- [10] B.Rusakov, S.Yankielowicz *Phys. Lett.* **339B**, 258,(1994)
- [11] V. Kazakov and I. Kostov *Nucl. Phys.* **B176**,199 (1980)
- [12] M.R.Douglas, Cargese '93, hep-th/9311130
- [13] P. Rossi, *Ann. Phys.* **132**, 463 (1981)
- [14] D.J.Gross, A.Matytsin *Nucl. Phys.* **B437**, 541 (1995)
- [15] A. Jevicki and B. Sakita, *Nucl. Phys.* **B165**, 511 (1980)